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Mean-Field Bounds and Correlation Inequalities

Alan D. Sokal[1]

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I prove a new correlation inequality for a class of N-component classical ferromagnets ($1 \leqslant N \leqslant 4$). This inequality implies that the correlation functions decay exponentially and the spontaneous magnetization is zero, above the mean-field critical temperature.

KEY WORDS: Mean-field theory; correlation inequalities; GHS inequality; Lebowitz inequality; critical temperature; Gaussian model.

Recently it has been shown, for a variety of classical lattice systems, that the mean-field critical temperature T_c^{MF} is a rigorous upper bound to the actual critical temperature T_c, in the sense that for $T > T_c^{\mathrm{MF}}$ the spontaneous magnetization is zero,[1-5] the correlation functions decay exponentially,[6-10] and there is a unique (regular) Gibbs state.[11-13] My purpose here is to show that, for a large class of N-component ferromagnets ($1 \leqslant N \leqslant 4$), all these properties are extremely simple consequences of a new (but not difficult) correlation inequality. This correlation inequality has also been used recently by Aizenman[14] as a lemma in one version of his proof of the Gaussianness of the continuum limit of φ_d^4 (or Ising_d) models in dimension $d > 4$.

Consider a finite one-component ferromagnet defined by the probability measure

$$d\mu(\varphi) = Z^{-1}\exp\left[(1/2)\sum_{i,j} J_{ij}\varphi_i\varphi_j + \sum_i h_i\varphi_i\right]\prod_i d\nu_i(\varphi_i) \tag{1}$$

with $J_{ij} = J_{ji} \geqslant 0$ and $h_i \geqslant 0$. Assume that each $d\nu_i$ is an even probability measure satisfying the hypotheses of the GHS and Lebowitz inequali-

[1] Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, New York 10012.

ties.[(15–19,10)2] Examples are the spin-$\frac{1}{2}$ Ising model

$$d\nu_i(\varphi) = \delta(\varphi^2 - 1)\, d\varphi \tag{2}$$

and the φ^4 lattice field theory

$$d\nu_i(\varphi) = \text{const} \times \exp(-a\varphi^2 - b\varphi^4)\, d\varphi \tag{3}$$

$(b > 0)$.

Consider first the case of zero magnetic field, i.e., all $h_i = 0$. Then I claim that the correlation functions of the model (1) are bounded above by the correlation functions of the *Gaussian* model with the same pair interaction J_{ij} and with single-spin measures having the same variance. That is, let

$$c_i = \int \varphi^2\, d\nu_i(\varphi) \tag{4}$$

and

$$d\nu_i'(\varphi) = (2\pi c_i)^{-1/2} \exp(-\varphi^2/2c_i)\, d\varphi \tag{5}$$

and let $d\mu'$ be the probability measure defined as in (1) but with $d\nu_i'$ replacing $d\nu_i$. Then we have the following theorem.

Theorem 1. Assume that all $h_i = 0$, and that each $d\nu_i$ satisfies the hypotheses of the zero-field Lebowitz inequality. Then, for each product of spins

$$\varphi^A = \prod_i \varphi_i^{A_i} \tag{6}$$

(here $A = \{A_i\}$ is a multi-index), we have

$$0 \leqslant \langle \varphi^A \rangle_\mu \leqslant \langle \varphi^A \rangle_{\mu'} \tag{7}$$

whenever the measure μ' is well defined.

Proof. $\langle \phi^A \rangle_\mu \geqslant 0$ is Griffiths' first inequality.[(16)] We next demonstrate that

$$\langle \varphi_i \varphi_j \rangle_\mu \leqslant \langle \varphi_i \varphi_j \rangle_{\mu'} \tag{8}$$

[2] The GHS inequality[(15–19)] states that $\langle \varphi_i; \varphi_j; \varphi_k \rangle \equiv \langle \varphi_i \varphi_j \varphi_k \rangle - \langle \varphi_i \varphi_j \rangle \langle \varphi_k \rangle - \langle \varphi_i \varphi_k \rangle \langle \varphi_j \rangle - \langle \varphi_j \varphi_k \rangle \langle \varphi_i \rangle + 2 \langle \varphi_i \rangle \langle \varphi_j \rangle \langle \varphi_k \rangle \leqslant 0$. The Lebowitz inequality in zero magnetic field[(15–19)] states that $\langle \varphi_i \varphi_j \varphi_k \varphi_l \rangle - \langle \varphi_i \varphi_j \rangle \langle \varphi_k \varphi_l \rangle - \langle \varphi_i \varphi_k \rangle \langle \varphi_j \varphi_l \rangle - \langle \varphi_i \varphi_l \rangle \langle \varphi_j \varphi_k \rangle \leqslant 0$. For both of these inequalities to be valid, it suffices[(18,19)] that $d\nu(\varphi) = \exp[-V(\varphi)]\, d\varphi$, where V is even and differentiable, with V' convex on $(0, \infty)$; or that $d\nu$ be a limit of such measures. Also, these inequalities hold for classical Ising models of arbitrary spin, by the "analog system" method of Griffiths.[(20)] Finally, the zero-field Lebowitz inequality (but *not* the GHS inequality) is valid under the weaker condition[(10)] that $d\nu(\varphi) = \exp[-f(\varphi^2)]\, d\varphi$, where f is convex on $(0, \infty)$.

for all i, j. To do this, consider the probability measure $d\mu_\lambda$ defined as in (1) but with λJ_{ij} replacing J_{ij}, with $0 \leqslant \lambda \leqslant 1$. Then

$$\frac{d}{d\lambda}\langle \varphi_i\varphi_j\rangle_{\mu_\lambda} = \frac{1}{2}\sum_{k,l} J_{kl}\langle \varphi_i\varphi_j; \varphi_k\varphi_l\rangle_{\mu_\lambda}$$
$$\leqslant \sum_{k,l}\langle \varphi_i\varphi_k\rangle_{\mu_\lambda} J_{kl}\langle \varphi_l\varphi_j\rangle_{\mu_\lambda} \tag{9}$$

by the Lebowitz inequality. (I use the notation

$$\langle \varphi^A; \varphi^B\rangle \equiv \langle \varphi^A\varphi^B\rangle - \langle \varphi^A\rangle\langle \varphi^B\rangle.) \tag{10}$$

Now (9) is a system of differential inequalities for the functions

$$G_{ij}(\lambda) \equiv \langle \varphi_i\varphi_j\rangle_{\mu_\lambda} \tag{11}$$

Since the right side of (9) is an increasing function of each element of the matrix G (in the relevant region $G_{ij} \geqslant 0$), it follows from a well-known lemma(21) that the solution of the differential inequality is bounded above by the solution of the corresponding differential *equation* with the same initial condition at $\lambda = 0$. This initial condition is

$$G_{ij}(0) = c_i\delta_{ij} \tag{12}$$

and the solution of the differential equation is

$$\bar{G}(\lambda) = \left[G(0)^{-1} - \lambda J\right]^{-1} \tag{13}$$

We thus have

$$G_{ij}(\lambda) \leqslant \bar{G}_{ij}(\lambda) \tag{14}$$

for all i, j. But notice now that $\bar{G}_{ij}(\lambda)$ is nothing other than the two-point function of the Gaussian model μ'_λ; in particular,

$$\bar{G}_{ij}(1) = \langle \varphi_i\varphi_j\rangle_{\mu'} \tag{15}$$

[This is easily seen by direct computation, or simply by noting that the Lebowitz inequality (9) becomes *equality* for the Gaussian model.] This proves (8).

To derive the general case (7), we now use the Gaussian inequality(22–24,10) (which is a consequence of the same hypotheses as the Lebowitz inequality(24,10)):

$$\langle \varphi^A\rangle_\mu \leqslant \sum_{\text{pairings}} \prod \langle \varphi_i\varphi_j\rangle_\mu \leqslant \sum_{\text{pairings}} \prod \langle \varphi_i\varphi_j\rangle_{\mu'} = \langle \varphi^A\rangle_{\mu'} \tag{16}$$

by the Gaussian inequality for μ, the inequality (8), and the Gaussianness of μ'. ■

Remarks. 1. The measure μ' is well defined only so long as the matrix

$$K = G(0)^{-1} - J \tag{17}$$

is strictly positive definite. Otherwise the solution (13) blows up before $\lambda = 1$.

2. The proof of Theorem 1 is very similar to the proof of Simon's inequality in intermediate-bond form (Ref. 8, Theorem 3.1). A significantly stronger version of Simon's inequality has been proven by Brydges, Fröhlich, and Spencer (BFS) (Ref. 10, Theorem 6.1) using entirely different methods. Tantalizingly, both this stronger inequality and Theorem 1 would follow immediately from the kind of argument used here, *if* it were true that any solution to the matrix differential inequality

$$0 \leqslant \frac{dG}{d\lambda} \leqslant GJG \tag{18}$$

$(G, J \geqslant 0)$ necessarily satisfies

$$G(\lambda) \leqslant G(0) + \lambda G(0) J G(\lambda) \tag{19}$$

(all inequalities interpreted elementwise). Unfortunately, this conjecture, while true for scalars, is *false* for matrices.[3] It would be interesting to find a proof of the BFS inequality using differential-inequality or duplicate-variable methods.

3. Theorem 1 carries over immediately to the infinite-volume limit for any boundary conditions (b.c.) which respect the requirements $J_{ij} \geqslant 0$ and $h_i = 0$. For example, periodic, Neumann, and zero ($\equiv$ Dirichlet $\equiv$ free) b.c. are all allowable; however, plus or minus b.c. are not.

We now apply Theorem 1 to a translation-invariant model with all $c_i = c$ and $J_{ij} = J(i-j)$. We define

$$\mathcal{J} = \sum_j J_{0j} \tag{20}$$

[3] A counterexample can be found by taking the Ansatz

$$G(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ b(\lambda) & a(\lambda) \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $a(0) = 1$, $b(0) = 0$. Inequality (18) requires that $0 \leqslant da/d\lambda \leqslant 2ab$ and $0 \leqslant db/d\lambda \leqslant a^2 + b^2$. We wish to violate inequality (19) either on-diagonal $[a(\lambda) \nleqslant 1 + \lambda b(\lambda)]$ or off-diagonal $[b(\lambda) \nleqslant \lambda a(\lambda)]$, for some $\lambda > 0$. This can be arranged by taking $a(\lambda) = 1 + \lambda^2 + \lambda^4 + B\lambda^5$ and $b(\lambda) = \lambda + \lambda^3 + D\lambda^4 + E\lambda^5 + F\lambda^6$. To satisfy (18) and violate (19) on-diagonal for λ small and positive, take $D < B < 2D/5 < 0$. To satisfy (18) and violate (19) off-diagonal for λ small and positive, take $D = 0$, $E = 1$ and $B < F < B/3 < 0$.

Then the norm of the matrix J, considered as an operator on l^2 (or on any l^p space, $1 \leqslant p \leqslant \infty$), is at most $\mathcal{J}$; so the Neumann series for $\bar{G}(1) = [G(0)^{-1} - J]^{-1}$ converges whenever

$$c\mathcal{J} < 1 \tag{21}$$

This is precisely the mean-field condition for the absence of a phase transition.[5,11,25] Moreover, it is easy to see that $\bar{G}(1)$ then decays exponentially if J does. For example, in the nearest-neighbor model with interaction strength β on the lattice Z^d (so that $\mathcal{J} = 2d\beta$), we have

$$\bar{G}(1)_{ij} \leqslant \text{const} \times \exp(-m_0|i-j|) \tag{22}$$

with mass gap

$$m_0 = \cosh^{-1}\left[(2\beta c)^{-1} - (d-1)\right] \tag{23}$$

(Note that $m_0 > 0$ precisely when $c\mathcal{J} \equiv 2d\beta c < 1$; this is because the mean-field critical temperature is exact for the Gaussian model.) Then Theorem 1 implies that (22) is an upper bound for the exact two-point function (in the non-Gaussian model μ), and hence the mass gap is at least that given by (23).

Remark. Strictly speaking, the above analysis should be carried through for each *finite* volume Λ, and then the inequalities carried over to the infinite-volume limit. But the result of such an analysis is obviously the one given above.

We can also show that $c\mathcal{J} < 1$ implies zero spontaneous magnetization and a unique Gibbs state (or more precisely, a unique *regular* Gibbs state if the single-spin measure $d\nu = d\nu_i$ has unbounded support). Indeed, the exponential (or even integrable) decay of the two-point function in the zero-b.c. infinite-volume state *implies* these two other properties, by an argument using the GHS and FKG inequalities.[26] The same conclusion can be obtained (by what is in essence the same argument) using an easy extension of Theorem 1 to the case $h_i > 0$:

Theorem 2. Assume that all $h_i \geqslant 0$ and that each $d\nu_i$ satisfies the hypotheses of the GHS and zero-field Lebowitz inequalities. Then, for all i, j,

$$0 \leqslant \langle \varphi_i \rangle_\mu \leqslant \langle \varphi_i \rangle_{\mu'} \tag{24}$$

and

$$0 \leqslant \langle \varphi_i ; \varphi_j \rangle_\mu \leqslant \langle \varphi_i ; \varphi_j \rangle_{\mu'} \tag{25}$$

whenever the measure μ' is well defined.

Proof. Let us fix J and imagine varying the magnetic fields $h = \{h_k\}$. We write

$$M_i(h) = \langle \varphi_i \rangle_{\mu(h)} \tag{26}$$

and

$$\begin{aligned} G_{ij}(h) &= \langle \varphi_i; \varphi_j \rangle_{\mu(h)} \\ &= \langle \varphi_i \varphi_j \rangle_{\mu(h)} - \langle \varphi_i \rangle_{\mu(h)} \langle \varphi_j \rangle_{\mu(h)} \end{aligned} \tag{27}$$

and similarly with primes for the quantities in the Gaussian model $\mu'(h)$. Then

$$\frac{\partial M_i}{\partial h_k} = G_{ik}(h) \tag{28}$$

by an easy computation, and

$$\frac{\partial G_{ij}}{\partial h_k} = \langle \varphi_i; \varphi_j; \varphi_k \rangle_{\mu(h)} \leqslant 0 \tag{29}$$

by the GHS inequality. Thus, by integrating (29),

$$G_{ij}(h) \leqslant G_{ij}(h = 0) \tag{30}$$

Inserting this in (28) and integrating again, we get

$$M_i \leqslant \sum_k G_{ik}(h = 0) h_k \tag{31}$$

Now by Theorem 1, $G_{ij}(h = 0)$ is bounded above by the corresponding quantity in the Gaussian model $\mu'(h = 0)$:

$$G_{ij}(h = 0) \leqslant G'_{ij}(h = 0) \tag{32}$$

Moreover, in the Gaussian model one has

$$G'_{ij}(h) = G'_{ij}(h = 0) \tag{33}$$

and

$$M'_i = \sum_k G'_{ik}(h = 0) h_k \tag{34}$$

Combining (30)–(34) completes the proof. ■

It follows from Theorem 2 that the spontaneous magnetization in the plus-b.c. state[4] for the model μ is bounded above by the spontaneous

[4] Plus b.c. denotes an external configuration which eventually dominates almost every (a.e.) configuration in every regular Gibbs state (but which does not grow *too* fast). In the case of spins supported on the bounded interval $[-M, M]$, it suffices to take a constant configuration $\varphi_i = M$; while for superstable unbounded spins, it suffices to take $\varphi_i = K(\log|i|)^{1/2}$ for a suitable constant K.[28]

magnetization in the state with the same b.c. for the Gaussian model μ'. But if $c\mathcal{J} < 1$, a direct computation shows that the spontaneous magnetization of the Gaussian model is zero in the infinite-volume limit; so the same goes for the original model μ. It then follows from the theorem of Lebowitz and Martin-Löf[27] (as amended by Lebowitz and Presutti[28] in the case of unbounded spins) that there is a unique (regular) Gibbs state.

Finally, it is worth noting that a version of Theorem 1 holds also for a class of N-component isotropic ferromagnets with $N = 2, 3, 4$. Indeed, consider the model

$$d\mu(\boldsymbol{\varphi}) = Z^{-1}\exp\left[(1/2)\sum_{i,j} J_{ij}\boldsymbol{\varphi}_i \cdot \boldsymbol{\varphi}_j\right]\prod_i d\nu_i(\boldsymbol{\varphi}_i) \tag{35}$$

where $\boldsymbol{\varphi}_i = (\varphi_i^{(1)}, \ldots, \varphi_i^{(N)})$ and $J_{ij} = J_{ji} \geqslant 0$. Assume that each $d\nu_i$ is a rotationally invariant measure of the form

$$d\nu_i(\boldsymbol{\varphi}) = \text{const} \times \exp(-a|\boldsymbol{\varphi}|^2 - b|\boldsymbol{\varphi}|^4)\, d\boldsymbol{\varphi} \tag{36}$$

$b > 0$ ($\boldsymbol{\varphi}^4$ model) or

$$d\nu_i(\boldsymbol{\varphi}) = \text{const} \times \delta(|\boldsymbol{\varphi}|^2 - 1)\, d\boldsymbol{\varphi} \tag{37}$$

(N-vector model). Then for $N = 2, 3, 4$ the Griffiths, Lebowitz, and Gaussian inequalities hold in the form[29–32,24]

$$\langle \varphi_A^{(1)}\varphi_B^{(2)} \cdots \varphi_G^{(N)} \rangle \geqslant 0 \tag{38}$$

$$\langle \varphi_A^{(\alpha)}\varphi_B^{(\alpha)} \rangle \geqslant \langle \varphi_A^{(\alpha)} \rangle \langle \phi_B^{(\alpha)} \rangle \tag{39}$$

$$\langle \varphi_A^{(\alpha)}\varphi_B^{(\beta)} \rangle \leqslant \langle \varphi_A^{(\alpha)} \rangle \langle \varphi_B^{(\beta)} \rangle \qquad \text{for } \alpha \neq \beta \tag{40}$$

$$\langle \varphi_A^{(\alpha)} \rangle \leqslant \sum_{\text{pairings}} \prod \langle \varphi_i^{(\alpha)}\varphi_j^{(\alpha)} \rangle \tag{41}$$

(Here we write $\varphi_A^{(\alpha)}$ instead of $\varphi^{(\alpha)A}$ to denote a product of spin components, for notational convenience.) For $N = 2$ these inequalities hold under the weaker assumption that $d\nu_i(\boldsymbol{\varphi}) = \exp[-f(|\boldsymbol{\varphi}|^2)]\,d\boldsymbol{\varphi}$ with f convex on $(0, \infty)$, or that $d\nu_i$ be a limit of such measures.[3,10] Then we have

Theorem 3. Consider the model (35) with $N = 2$, 3, or 4 and all $d\nu_i$ as above, and let μ' be the corresponding N-component isotropic Gaussian model. Then, for each product $\varphi_A^{(\alpha)}$, we have

$$0 \leqslant \langle \varphi_A^{(\alpha)} \rangle_\mu \leqslant \langle \varphi_A^{(\alpha)} \rangle_{\mu'} \tag{42}$$

whenever the measure μ' is well defined.

The proof is virtually identical to that of Theorem 1, employing (38) in place of the Griffiths inequality, (40) and (41) in place of the Lebowitz inequality, and (41) in place of the Gaussian inequality.

Theorem 3 implies exponential decay of the correlation functions (of a single-spin component $\varphi^{(\alpha)}$) for $T > T_c^{\mathrm{MF}}$ and $N = 2, 3, 4$, by an argument identical to that used for $N = 1$. Unfortunately, the N-component analog of Theorem 2 is problematic, since no sharp analog of the GHS inequality has yet been proven for multicomponent spins.[24,31] Nevertheless, the vanishing of the spontaneous magnetization for $T > T_c^{\mathrm{MF}}$ in the N-vector model for arbitrary N has been shown by Simon[4] using a correlation inequality of Ginibre type. For $N = 2$ this implies that all Gibbs states are rotation-invariant[33] and that there is a unique translation-invariant Gibbs state.[33,34]

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NOTE ADDED IN PROOF

References 35 and 36 contain additional material related to mean-field bounds. In particular, a result slightly *stronger* than Theorem 1 is proved in reference 36 for the *special case* of the spin-$\frac{1}{2}$ Ising model. The proof uses Griffiths' third inequality [1] combined with the Krinsky–Emery–Simon[6,8] iteration argument. I thank Professor Joel Lebowitz for calling these references to my attention.

Brydges[37] has shown that a generalized form of Simon's inequality is an easy consequence of the strong Gaussian inequality[22,23,10]. Thus (19) holds for the spin systems considered here, but it cannot be proved by our differential-inequality arguments!

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